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EXPONENTIAL UTILITY MAXIMIZATION OF AN INVESTOR'S STRATEGY USING MODIFIED CONSTANT ELASTICITY OF VARIANCE AND ORNSTEIN-UHLEMBECK MODELS

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ABSTRACT

This work is aimed at finding the optimal investment strategy for an investor under the modified constant elasticity of variance (M-CEV) and Ornstein-Uhlenbeck models. We assume that the stock price is governed by modified constant elasticity of variance (M-CEV) model, where the investor has an exponential utility preference. We also investigate the impact of the correlation of the Brownian motions. Dynamic programming principle, precisely, the maximum principle is applied to obtained the Hamilton-Jacobi-Bellman (HJB) equation, on which elimination of variable dependency was applied to obtain the closed from solution of the optimal investment strategies. It was verified that the investor's optimal investment strategy when the Brownian motions correlate is greater than the investor's optimal investment strategy when the Brownian motions do not correlate by a fraction, $\left\{\frac{\varphi\delta a^2 S^{(2\gamma+1)}+\varphi\delta}{f a^2 S^{2\gamma}}\right\}$.

Keywords: Optimal investment strategy, investor, modified constant elasticity of variance (M-CEV), ornstein-uhlenbeck model, exponential utility maximization.

2010 Mathematics Subject Classification: 91G10, 91G80

INTRODUCTION

Among the many problems researched into in financial mathematics and actuarial science found in literature is financial asset allocation problems in discrete and continuous time. The origin of this can be traced to Merton, Merton (1969) and Merton (1971).

Merton, 1971 solved the problem of an investor whose goal was to maximize his expected utility when investing in stock and consumption is allowed where the underlying asset was modeled by Black-Scholes model with a given utility preference. Markowitz's work gave rise to modern portfolio theory which is based on the assumption that the investing public's efforts are geared towards the minimization of risk to gain the possible optimum return. He showed that different portfolios with varied levels of risk and return helped an investor to decide the level of risk he could accommodate to enable diversification of his portfolios. It is expected that the investor is rational and could react within these premise taking decisions which is hoped to maximize his return under a certain level of uncertainty. The volatility the investor would accept in his alternative portfolio is done by making a selection that is parallel to the efficient frontier that would enable him achieve a maximum return for the quantum of risk he has assumed. One of the ways the examination of optimal portfolios may be complemented is by approximating different expected value of returns for a number of times for every risk taken. In the theory of interest mathematics and optimal portfolio selection, utility maximization is given proper attention as it a mode of assessing ones satisfaction for making an investment. Based on this, many scholars have been motivated to analyze investment problems using stochastic control technique.

Karatzas *et al*. (1987) used the martingale technique to solve investment problems bothering on utility maximization where the price process of the risky asset followed the geometric Brownian motion for an incomplete market. This implies that the volatility of the risky asset is deterministic which empirical evidence did not supported, hence it become apparent that the use of stochastic volatility model is assumed to be more realistic.

An extension of the geometric Brownian motion model is the constant elasticity of variance model which is a stochastic volatility formula. It is used to capture the implied volatility. Gao (2009) used this constant elasticity of variance model to model life insurance annuity policies

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and the optimal investment strategies with utility function using dynamic programming principle. He studied the problem of an investor who wishes to maximize the expected utility of terminal wealth when he can invest in both a risk-less asset and a risky asset. He applied the stochastic optimal control, the Hamilton-Jacobi-Bellman (HJB) equation and the maximum principle to transform a complex non-linear partial differential equation to a simplified partial differential equation from which he obtained an explicit solution to the investor's problem.

Another study Liu *et al*. (2012) solved a problem of utility maximization by transforming a non-linear second-order partial differential equation into a linear partial differential equation using the method of elimination of dependency on the wealth variable, w, and the price variable, π, of the risky asset. They demonstrated this within the context of continuous time framework where the state variable in the stochastic differential equation is wealth and the controls are the assets' shares. They found that the determinants of the change in wealth are the stochastic returns on assets and the interest rate on the risk-less asset for the given asset allocation at a time.

This research on exponential utility maximization of an investor's strategy using modified constant elasticity of variance and Ornstein-Uhlenbeck model takes into consideration the cases of when the Brownian motions correlate and when they do not correlate. We wish to unravel the impact of correlation of the Brownian motions on the optimal investment strategy of an investor with the modified constraint elasticity of variance using the Ornstein-Uhlenbeck models.

We have cited some works we went through to initiate this work. Some other works reviewed include; Oksendal *et al*. (2002) who investigated a market with one risk–free and one risky asset in which the dynamics of the risky assets price $S(t)$ is governed by a Geometric Brownian Motion. They considered an investor who assumes a bank account that has the opportunity at any time to transfer funds between two assets and assume that these transfers involve a fixed transaction cost which was independent of the size of the transaction plus its cost proportional to the size of the transaction. Their formulated the problem as a combine stochastic control and impulse control type and maximized the cumulated expected utility of consumption over the planning horizon.

Gu and Gao (2012) obtain optimal strategies and optimal value functions under constant elasticity of variance (CEV) model on the condition that the insurer could purchase excess-of-loss reinsurer. Merton (1971) worked on continuous time allocation problem under uncertainty. He considered a model in which the processes of the risky assets are generally correlated to geometric Brownian Motion (GBM) and assumes that the portfolio can rebalance instantly with no cost. His maximized the net expected utility of consumption and the utility of terminal wealth in order to keep the proportions invested in the risky assets equal to a constant vector and to a consumer at a rate that is proportional to the total wealth. He consider an optimal trading strategy which consisted of an infinite number of transactions and utility function in the constant relative risk aversion (CRRA) case.

Another study Gu (2010) worked on the optimal investment maximization using the constant elasticity of variance (CEV) model. Also Lin and Li (2011) considered an optimal reinsurance investment problem for an insurer with jump diffusion risk process under the constant elasticity variance (C.E.V.) model. Zhao and Rong (2012) studied the portfolio selection problem with multiple risky assets under the constant elasticity of variance (CEV) model. They obtained a strategy that is suitable to the investor's problem. Guo *et al*. (2012) studied the optimization of define contribution (DC) pension scheme under a constant elasticity of variance model and obtained the closed from solution of the optimal investment strategy for power and exponential utility functions. Chang *et al*. (2013) considered an asset and liability management problem with stochastic interest rate assuming affine interest rate model. Ihedioha (2016) considered exponential utility optimization of an investor's optimal portfolios, under constant elasticity of variance model.

Ihedioha *et al*. (2017) investigated the effect of correlation of Brownian motions on an investor's optimal investment and consumption decision under the Ornstein-Uhlenbeck model. The aim of their work was to obtain a closed-form solution to the investment and consumption problem where the risk-free asset has a rate of return that is driven by the Ornstein-Uhlenbeck stochastic model. The maximum principle is applied to obtain the HJB equation for the value function. Owing to the introduction of the consumption factor and the Ornstein -Uhlenbeck stochastic interest rates of return, the HJB equation derived became much more difficult and they employed the method elimination of dependency on variables to transformed the non–linear second-order partial differential equation to ordinary differential equation, specifically the Bernoulli equation which they solved to obtain the investor's optimal strategy and value function..

In a recent year Ihedioha (2017) studied "optimal asset allocation problem for an investor with Ornstein-Uhlenbeck stochastic Rate Model. Also Ihedioha *et al*. (2017) considered the impact of consumption on an investor's strategy under Ornstein-Uhlenbeck model: The case of Non-correlating Brownian motions. Health and Platen (2002) developed a consistent pricing and hedging process for a modified constant Elasticity of variance model (M-CEV). Their study considered a modification of the popular constant elasticity of variance model which is used to model the growth optimal portfolio. They proved that there is no equivalent risk or neutral pricing measure and therefore, the classical risk neutral pricing methodology fails. The research showed that a consistent pricing and hedging framework can be established by application of the bench mark approach. The model assumes a bench mark approach using the GOP as benchmark. The derived benchmark provides a consistent pricing and hedging framework without requiring the existence of an equivalent risk neutral measure. This gives rise to a hedge ratio that describes the number of units of the savings account at a time that has to be held in a corresponding self-financing hedge portfolio. and consequently, Combining the self-financing Hedge portfolio's developed and the discounted cost for maintaining the hedge portfolio a differential equation is obtained which shows that the discounted cost is also constant and equal to the initial fair price for the contingent claim. The work further shows that benchmark and nonnegative price process replicate the contingent claim.

Muravey (2017) studied optimal investment problem with modified constant elasticity of variance (M-CEV) model and obtained closed-form solutions and application to the algorithmic tradition. The study took an optimal utility function used Laplace transforms to obtain an explicit expression for the optimal strategy in terms of confluent hyper-geometric functions. For the representations obtained, the asymptotic and approximation formulas are derived, containing only elementary functions.

MATERIALS AND METHODS

For better understanding of this work we present in brief the following concepts:

BROWNIAN MOTION

Brownian motion is seen as a simple continuous stochastic process that is widely used in physics, Chemistry and in the financial world for modeling random movement of molecules of gas or fluctuation in an assets price.

In mathematics and economics, Brownian motion is described as a continuous time stochastic process called wiener process, named in honor of Norbert Wiener.

The Wiener process W(t) is characterized by four facts as follows:
(i). $Z_0 = 0$. (ii). Z_1 is almost surely continuous. (iii).Z_t has independent increments. (iv). $Z_t - Z_s \sim N(0, t-s)$ (for $0 \leq s \leq t$).
$N(\mu, \delta^2)$ denotes the normal distribution with the expected value μ and variance δ^2. The condition that it has independent increment means that if $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$ then $Z_{t_1} - Z_{s_2}$ are independent random variables.

ORNSTEIN – UHLENBECK MODEL

There are several approaches to model interest rate and commodity price stochastically. One of the ways is the Ornstein – Uhlenbeck model (perhaps with modification). An Ornstein–Uhlenbeck process r(t), satisfies the following stochastic differential equation:

$$dr(t) = \theta(\mu - r(t))dt + SdW(t)$$

Where θ>0, μ and $\delta > 0$ are parameters and $dW(t)$ denotes the wiener process. We can also call this vasicek model.

CONSTANT ELASTICITY OF VARIANCE (CEV) MODEL

This is a model used in financial mathematics to model stochastic volatility. In an attempt to capture stochastic volatility, market shocks and the leverage effects, to this end, the model is widely used by practitioners in the financial industry for modeling of equities and commodity prices. It was developed by John Cox in 1975. The constant Elasticity of variance model (process) and is analytically traceable, leading to closed–form options pricing formulas. The constant elasticity of variance (C.E.V) model is a one-dimensional diffusion process that solves a stochastic differential equation
$dS(t) = \mu S(t)\, dt + \delta S(t)\gamma dW(t)$,where S(t) is the spot price, t is time and μ is a parameter characterizing the drift, the instantaneous validity δ(S) = $\delta S^{\gamma}(t)$, $\delta\ and\ \gamma$ are also parameters and W(t) is a Brownian motion. The difference "$dS(t)$" represents an infinitesimal small change in parameter S.
The constant $\delta \geq 0$, and the parameter γ, which is the central feature of the model, controls the relationship between volatility and the price of the asset. If $\gamma < 1$, we have the leverage effect, which is commonly observed in equity markets where stocks volatility increases as the time decreases. As in commodity markets, γ>1, we have inverse leverage effect in which case the volatility of the price of a commodity increase as its price increases.

MODIFIED CONSTANT ELASTICITY OF VARIANCE MODEL (M- CEV)

A lot of efforts have been made on optimal investment problems with the assumption on that the price follows geometric Brownian motion (GBM). However, there are a lot of empirical studies showing this simple model does not properly fit to real market data. Known draw backs are the following:

(i) GBM model does not properly capture volatility smile/skew effects.
(ii) They ignore the probability the underlying default
(iii) The constant coefficients do not allow calibration of this model to the real term structure of interest rates and dividend yields etc.

The (M–CEV) model aims to extend the results of the GBM models to a more realistic model.

We choose this model for the following reasons
(a) The model captures the volatility smile effects
(b) It allows non–zero probability of the underlying's default (M- C.E.V process can touch zero while GBM is always positive)
(c) It is analytically tractable.
(d) The model applicable to algorithmic trading strategies
(e) For the M–CEV model, we obtain a close form solution in terms of confluent hyper geometric functions.

This model was introduced by Heath and Platen (2002). It was applied to the algorithmic trading.

DYNAMIC PROGRAMMING

Dynamic programming or recursive optimization is a technique used for obtaining solutions for multi-stage decision problems. In general, there is no standard mathematical formulation of the variable given and the objective of the problem, an equation is developed to fit for a particular solution. In today's world, application of dynamic programming has become sacrosanct in our day-to-day administrative/managerial problems such as resource allocation, inventory problems logistics etc. Dynamic programming may be classified depending on the nature of data at hand as deterministic and stochastic or probabilistic models. For the deterministic models, the outcome at any decision stage is uniquely determined and known. The technique was developed by Richard Bellman in the early (1950s).

Principle of Optimality: It implies that a wrong decision taken at an earlier stage does not prevent from taking an optimal decision for the remaining stages. Thus, this principle is a firm base for dynamic programming technique

MAXIMUM PRINCIPLE

In order to find the best control for taking a dynamic system from stage to stage sequentially in the presence of inhibitors (constraints) or input controls, we apply the maximum principle which is a powerful tool in optimal control theory.

It was Lev Pontryagin, a Russian mathematician, in collaboration with his students that formulated this principle, which uses Euler-Lagrangian equation of calculus of variation. This principle in an informal way states that the control Hamiltonian takes extreme value in all admissible control set. The control, minimum or maximum has a dependency on the problem and the sign convention adopted in defining the Hamiltonian. It is called maximum principle because normal convention has it that the Hamiltonian that is used will lead to a maximum.

Let U be the permissible control set of values, the maximum principle states that if the control U^*satisfies:$H(x^*(t), u^*(t), \lambda^*(t), t) \leq H(x^*(t), u(t), \lambda^*(t), t),; u \epsilon U, t \epsilon [t_o, t_f]$,such that, $x^* \epsilon C^1[t_o, t_f]$, $x^* \in c^1[t_0, t_f]$ is the optimal co-state trajectory (a special type of optimization problem where the decision variables are functions rather than real numbers) and $\lambda^* \in B \cup [t_0, t_f]$is the optimal co-state trajectory, then U^* is the optimal control.

HAMILTON–JACOBI–BELLMAN (HJB) EQUATION

Are second orders, degenerate, elliptic, non-linear partial differential equation which are central to optimal control theory. The solution of the HJB equation is the cost for a given dynamical system with an associated cost function.

When solved locally, the Hamilton–Jacobi Bellman (HJB) equation is a necessary condition, but when solved over the whole state space, the HJB equation is a necessary and sufficient condition for an optimum. The solution is open loop, but it also permits the solution of the closed loop problem. The Hamilton-Jacobi-Bellman equation can be generalized to stochastic system as well. The equation is a result of the theory of dynamic programming which was pioneered by Richard Bellman and his co-workers.

UTILITY FUNCTION AND EXPONENTIAL UTILITY FUNCTIONS

Utility is an economic term introduced by the noted 18th century Swiss mathematician Daniel Bernoulli, referring to the total satisfaction received from consuming a good or service. It could also mean a function that specifies the wellbeing of a consumer for all combinations of goods or services consumed.

The exponential utility function refers to a specific form of utility function that is sometimes referred to as uncertainty. It is used in some contexts because of its convenience when risk is present, in which case expected utility is maximized. Thus the exponential utility function is given by:

$$u(v) = \frac{(1-e^{-fv})}{f}, \quad f \neq 0.$$

V is the variable that the economic decision maker prefers more of such as consumption, and f is a constant that represents the degree of risk preference (f>0 for risk aversion, f=0 for risk neutrality, or f<0 for risk seeking).
In a situation where only risk aversion is allowed, the formula is often simplified to $u(v) = (1 - e^{-fv})$.

MODEL FORMULATION AND THE MODEL

Assuming that an investor trades two assets in the financial market; a risky asset (stock) and a riskless (risk free) asset (bond) that has a rate of return that is a function of time. The dynamics of the price of the risk-free asset denoted by $B(t)$ is $\frac{dB(t)}{B(t)} = r(t)dt$ (1)

The risky asset is governed by the modified constant elasticity of variance (M- CEV), model stated as follows;
$$\frac{dS(t)}{S(t)} = [\mu + Ca^2S^{2\gamma})dt + aS^\gamma dW^1(t)] \qquad (2)$$

where S(t) denotes the price of the risky asset at time t. c, a are constants, μ is the appreciation rate of the risky asset,$\{W(t): t > 0\}$ is a standard Brownian motion in a complete probability space $(\Omega, F, (F)t > 0, p)$. $(F)t > 0$ is the augmented filtration generated by the Brownian motion $W(t)$, $aS^\gamma(t)$ is the instantaneous volatility and the elasticity γ, a parameter for discount factor which satisfies the general condition $\gamma \leq 0$. If the elastic parameter $\gamma = 0$, then the modified constant elasticity of variance (M-CEV) model reduces to geometric Brownian motions.

The Ornstein-Uhlenbeck process is one of several approaches used to model (perhaps with modifications) interest rates, currency exchange rates and commodity prices stochastically. It is given as;
$$dr(t) = \varphi(\mu - r(t))dt + \delta dW^2(t). \qquad (3)$$

Let $K(t)$ be the amount of money the investor puts into the risky asset at time t, and then $[V(t) - K(t)]$ is the amount of money invested in the risk-free asset, where $V(t)$ is the total money invested in both assets. The dynamics of the wealth process corresponding to the trading strategy $K(t)$, is the stochastic differential equation (SDE)
$$dV(t) = K(t)\frac{dS(t)}{S(t)} + [V(t) - K(t)]\frac{dB(t)}{B(t)}, \quad (4)$$

Substituting (1) and (2) into 4) we get
$$dV(t) = \{(\mu - r(t))K(t) + r(t)V(t)\}dt + K(t)Ca^2S^{2\gamma}(t) + K(t)aS^\gamma dW^{(1)}(t)\}. \qquad (5)$$

The quadratic variation of equation (5) is
$$(dV(t))^2 = a^2S^{2\gamma}(t)K^2(t)dt. \qquad (6)$$

Note that at
$$\{dt.d\,W^{(1)}(t)) = dt.dt = 0, and\ d\,W^{(1)}(t).d\,W^{(1)}(t) = dt\}. \qquad (7)$$
Suppose an investor has an exponential utility function $U(V(t))$, then the investor's problem is to find the optimal strategy for
$$G(V,t;T) = Max_{K(t)}E[U(V(t))] \qquad (8)$$

subject to equation (5) $dV(t) = \{(\mu - r(t))K(t) + r(t)V(t)\}dt + K(t)Ca^2S^{2\gamma}(t) + K(t)aS^\gamma dW^{(1)}(t)\}$.

THE OPTIMIZATION

This study assumes that the investor has an exponential utility preference that is;
$$U(V) = -ae^{-fV}, \qquad (9)$$

with absolute risk aversion,
$$-\frac{U''(V)}{U'(V)} = -f. \qquad (10)$$
.We consider the cases;

CASE 1: When the Brownian motions do not correlate. (That is $E(W^{(1)}.W^{(12)}) = 0$

The Bellman equation corresponding to the investor's problem is
$$Max_{K(t)}E[G(V',t;T) - G(V,t;T)] = 0 \qquad (11)$$

where V' denotes the wealthy process at time $t + \Delta t$.
The division of both sides of (11) by Δt and taking limit as Δt tends to zero, we get
$$\text{Max}_{k(t)}\frac{1}{dt}E[dG] = 0. \qquad (12)$$

The maximum principle states that;
$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial S}dS + \frac{\partial G}{\partial r}dr + \frac{\partial G}{\partial V}dV + \frac{\partial^2}{\partial S \partial r}(dSdr) + \frac{\partial^2 G}{\partial S \partial V}(dSdV) + \frac{\partial^2 G}{\partial r \partial V}(drdV) + \frac{1}{2}\left[\frac{\partial^2 G}{\partial S^2}(dS)^2 + \frac{\partial^2 G}{\partial r^2}(dr)^2 + \frac{\partial^2 G}{\partial v^2}(dV)^2\right]. \qquad (13)$$

But,

$$\left.\begin{aligned}
dV(t) &= \{\{(\mu - r(t))K(t) + r(t)V(t) + k(t)Ca^2S^{2\gamma}(t)\}dt \\
&\quad + aS^{\gamma}(t)K(t)dW_1(t), \\
dr(t) &= \theta(\tau - r(t))dt + \delta dW_2(t), \\
dS(t) &= S(t)[\mu + Ca^2S^{2\gamma}(t)]dt + aS^{\gamma+1}(t)dW_1(t), \\
(dS(t))^2 &= a^2S^{2(\gamma+1)}(t)dt, \\
(dr(t))^2 &= \delta^2 dt, \\
(dV(t))^2 &= a^2S^{2\gamma}(t)K^2(t)dt, \\
(dS(t)dV(t)) &= a^2S^{2\gamma}(t)S(t)K(t)dt, \\
(dr(t)dV(t)) &= 0, \\
(dS(t)dr(t)) &= 0,
\end{aligned}\right\} \quad (14)$$

where

$$\left.\begin{aligned}
dW_1(t).dW_2(t) = dt.dt = dt.dW_1(t) = dt.dW_2(t) = 0, \\
dW_1(t).dW_1(t) = dW_2(t).dW_2(t) = dt
\end{aligned}\right\} \quad (15)$$

Using (14) and (15) in (13), we obtain

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial S}\{S(t)[\mu + Ca^2S^{2\gamma}(t)]dt + aS^{\gamma+1}(t)dW^{(1)}(t)\} + \frac{\partial G}{\partial r}\left\{\begin{matrix}\theta(\tau - r(t))dt \\ +\delta dW^{(2)}(t)\end{matrix}\right\} + \frac{\partial G}{\partial V}\left\{\left[\begin{matrix}(\mu - r(t))K(t) + r(t)V(t) \\ +k(t)Ca^2S^{2\gamma}(t)\end{matrix}\right]dt\ aS^{\gamma}(t)K(t)dW^{(1)}(t)\right\} + \frac{\partial^2 G}{\partial S\partial V}\{a^2S^{2\gamma}(t)S(t)K(t)dt\} + \frac{\partial^2}{\partial r\partial v}\{\phi\delta aS^{\gamma}(t)K(t)dt\} + \frac{1}{2}\left[\frac{\partial^2 G}{\partial S^2}\{a^2S^{2\gamma+1}(t)dt\} + \frac{\partial^2 G}{\partial r^2}\{\delta^2 dt\} + \frac{\partial^2 G}{\partial V^2}\{a^2S^{\gamma}(t)K^2(t)dt\}\right]. \quad (16)$$

Substituting (16) into equation (12) and taking expectation we get

$$G_t + G_s[\mu S + Ca^2S^{2\gamma+1}] + G_r[\theta(\tau - r)] + G_v[(\mu - r)k + rv + kca^2S^{2\gamma}] + G_{sv}[a^2S^{(2\gamma+1)}k] + G_{ss}\left[\frac{a^2S^{2\gamma+1}}{2}\right] + \frac{G_{rr}\delta^2}{2} + \frac{G_{vv}[a^2S^{2\gamma}K^2]}{2} = 0, \quad (17)$$

where

$$E[dW^{(1)}(t)] = E[dW^{(2)}(t)] = 0. \quad (18)$$

G_t, G_s, and G_r are first partial derivatives o G and G_{sv}, G_{ss},and G_{vv} are second partial derivatives.

Differentiating (17) with respect to $k(t)$, we have;

$$G_v[(\mu - r) + ca^2S^{2\gamma}] + G_{sv}(a^2S^{(2\gamma+1)}) + G_{vv}(a^2S^{2\gamma}K(t)) = 0. \quad (19)$$

Now, solving for $K(t)$ in equation (19) we obtain the optimal investment strategy in the risky asset as

$$K^*(t) = \frac{-[(\mu - r) + ca^2S^{2\gamma}]G_v}{[a^2S^{2\gamma}]G_{vv}} - \frac{G_{sv}[a^2S^{2\gamma+1}]}{G_{vv}[a^2S^{2\gamma}]}. \quad (20)$$

Rewriting equation (20) we have

$$K^*(t) = -\left[\frac{(\mu - r(t))G_v}{(a^2S^{2\gamma}(t))G_{vv}} + \frac{CG_v}{G_{vv}} + \frac{SG_{sv}}{G_{vv}}\right]. \quad (21)$$

Let

$$G(t, r, s, v) = h(t, s, r)[-ae^{-fv}] \quad (22)$$

be a solution to equation (19), such that at the terminal time T

$$h(t, s, r) = 1, \quad (23)$$

then

$$\Rightarrow \quad G_t = h'[-ae^{-fv}], G_s = h_s[-ae^{-fv}], G_r = h_r[-ae^{-fv}], G_v = h_v[afe^{-fv}], G_{sv} = h_v[afe^{-fv}], G_{rv} = h_r[afe^{-fv}], G_{ss} = h_{ss}[ae^{-fv}], G_{rr} = h_{rr}[ae^{-fv}], G_{vv} = h_{vv}[-af^2e^{-fv}] \quad (24)$$

Applying (22) and (24) in (21) and simplifying, we obtain

$$K^*(t) = \left[\frac{(\mu - r) + Ca^2S^{2\gamma} + Sa^2S^{(2\gamma+1)}}{a^2S^{2\gamma}f}\right]. \quad (25)$$

CASE II: when the Brownian motions correlate ($E[dW^{(1)}.dW^{(2)}] = \varphi dt$)

In this case equation (14) becomes

$$\left.\begin{aligned}
dV(t) &= \{\{(\mu - r(t))K(t) + r(t)V(t) + k(t)Ca^2S^{2\gamma}(t)\}dt \\
&\quad + aS^{\gamma}(t)K(t)dW_1(t), \\
dr(t) &= \theta(\tau - r(t))dt + \delta dW_2(t), \\
dS(t) &= S(t)[\mu + Ca^2S^{2\gamma}(t)]dt + aS^{\gamma+1}(t)dW_1(t), \\
(dS(t))^2 &= a^2S^{2(\gamma+1)}(t)dt, \\
(dr(t))^2 &= \delta^2 dt, \\
(dV(t))^2 &= a^2S^{2\gamma}(t)K^2(t)dt, \\
(dS(t)dV(t)) &= a^2S^{2\gamma}(t)S(t)K(t)dt, \\
(dr(t)dV(t)) &= \varphi\delta aS^{\gamma}(t)K(t)dt, \\
(dS(t)dr(t)) &= \varphi\delta aS^{2\gamma}(t)S(t)K(t)dt,
\end{aligned}\right\} \quad (26)$$

Also, substituting equation (26) into equation (13), and going through the procedures we get the new Hamilton-Jacobi-Bellman equation

$$G_t + G_s[\mu S + Ca^2S^{2\gamma}] + G_r[\theta(\tau - r)] + G_v[(\mu - r)K + rv + KCa^2S^{2\gamma}] + G_{sv}[a^2S^{2\gamma+1}K] + G_{sr}[\varphi\delta aS^{2\gamma+1}k] + G_{rv}[\varphi\delta aS^{\gamma}K] + \frac{1}{2}[G_{ss}[a^2S^{2\gamma+1}] + G_{rr}[\delta^2] + G_{vv}[a^2S^{2\gamma}K^2]]\} = 0. \quad (27)$$

where (18) holds.

Now the differentiation of (27) with respect to $K(t)$ and simplifying gives the optimal strategy for the investor as we obtain.

$$K^*(t) = -\left[\frac{(\mu-r)G_v}{(a^2S^{2\gamma})Gvv} + \frac{(C)G_v}{G_{vv}} + \frac{(S)G_{sv}}{G_{vv}} + \frac{(\varphi\delta S)G_{sr}}{G_{vv}} + \frac{(\varphi\delta)G_{rv}}{(aS^{\gamma})G_{vv}}\right]. \quad (28)$$

Now applying (24) through (27) and simplifying the investor's optimal strategy under exponential utility maximization as

$$K^*(t) = -\left[\frac{(\mu-r) - Ca^2S^{2\gamma} - a^2S^{2\gamma+1} - \varphi\delta S^{2\gamma+1} - \varphi\delta hr}{-fa^2\delta^{2\gamma}}\right]. \quad (29)$$

To do away with the dependency on $r(t)$, we let

$$h(t,s,r) = W(t,S)[-ae^{-fr}], \quad (30)$$

such that at the terminal time T,

$$W(T,S) = \frac{-e^{fr}}{a}. \quad (31)$$

From (30) we see that

$$h_r = W(t,S)[afe^{-fr}]. \quad (32)$$

Applying (32) in (29) and simplifying we obtain the investor's optimal strategy whet he Brownian motions correlate as

$$K^*(t) = \left[\frac{(\mu-r) + Ca^2S^{2\gamma} + a^2S^{(2\gamma+1)} + \varphi\delta}{fa^2S^{2\gamma}}\right]. \quad (33)$$

THE EFFECT OF THE CORRELATION OF BROWNIAN MOTIONS

The optimal investment strategy when the Brownian motions do not correlate is given by

$$K^*_{nc}(t) = \left[\frac{(\mu-r) + a^2S^{(2\gamma+1)} + Ca^2S^{2\gamma}}{fa^2S^{2\gamma}}\right],$$

as in equation (25), and when the Brownian motions correlates by

$$K^*_c(t) = \left[\frac{(\mu-) + Ca^2S^{2\gamma} + a^2S^{(2\gamma+1)} + \varphi\delta a^2S^{(2\gamma+1)} + \varphi\delta}{fa^2S^{2\gamma}}\right],$$

as in equation (33). We deduce from these equations that

$$K^*_c(t) = \left\{\frac{(\mu-r) + a^2S^{(2\gamma+1)} + Ca^2S^{2\gamma}}{fa^2S^{2\gamma}}\right\}, + \left\{\frac{\varphi\delta a^2S^{(2\gamma+1)} + \varphi\delta}{fa^2S^{2\gamma}}\right\},$$

and

$$k^*_c(t) = K^*_{nc}(t) + \varphi\left\{\frac{\delta a^2S^{(2\gamma+1)} + \delta}{fa^2S^{2\gamma}}\right\}. \quad (35)$$

We now consider the following four cases for the correlation;

CASE 1: When the correlation is negative:

If

$$\varphi = -m, \quad (36)$$

we have from equation (35)

$$k^*_c(t) = k^*_{nc}(t) - m\left\{\frac{\delta a^2S^{(2\gamma+1)} + \delta}{fa^2S^{2\gamma}}\right\}. \quad (37)$$

This implies that, the investor's optimal investment strategy when the Brownian motions correlate negatively is less than the investor's optimal investment strategy when the Brownian motion do not correlate by a fraction, $\frac{m\delta(a^2S^{(2\gamma+1)}+1)}{fa^2S^{2\gamma}}$.

CASE II: When the correlation is be positive

If

$$\varphi = m. \quad (38)$$

we have

$$k^*_c(t) = k^*_{nc}(t) + m\delta\left\{\frac{a^2S^{(2\gamma+1)} + 1}{fa^2S^{2\gamma}}\right\}. \quad (39)$$

In this case, the investor's optimal investment strategy when Brownian motions correlate positively is greater than the investor's optimal investment strategy when the Brownian motions correlate by the fraction, $\frac{m\delta(a^2S^{(2\gamma+1)}+1)}{fa^2S^{2\gamma}}$, where m is the correlation coefficient. This shows that the investor will require more amount of money to invest in the risky asset.

CASE III: When the correlation is unity;

If $\varphi = 1, \quad (40)$

we have

$$K^*_c(t) = K^*_{nc}(t) + \delta\left\{\frac{a^2S^{(2\gamma+1)} + 1}{fa^2S^{2\gamma}}\right\}. \quad (41)$$

This implies that, the investor's optimal investment strategy when the Brownian motion correlate is less than the investor's optimal investment strategy when the Brownian motions do not correlate by $\delta\left\{\frac{a^2S^{(2\gamma+1)}+1}{fa^2S^{2\gamma}}\right\}$.

CASE IV: When the correlation is equal to zero.

If

$$\varphi = 0, \quad (42)$$

We have

$$K^*_c(t) = K^*_{nc}(t)\,. \quad (43)$$

In this case the investor's optimal investment strategy when the Brownian motion correlate is equal to the investor's optimal investment strategy when the Brownian motions do not correlate.

CONCLUSION

From Equation (35)

$$K^*_c(t) = K^*_{nc}\,(t) + \phi\delta\left\{\frac{a^2S^{(2\gamma+1)} + 1}{fa^2S^{2\gamma}}\right\},$$

we observe that when the Brownian motion correlate, the optimal strategy is greater than when the Brownian motions do not correlate by a fraction $\frac{\phi\delta\{a^2S^{2\gamma+1}\}}{fa^2S^{2\gamma}}$.

Also we obtained the effects of the correlation which were in the following four perspectives:

i. When correlation is negative, then the strategy reduces by a fraction: $\frac{m\delta(a^2S^{(2\gamma+1)}+1)}{fa^2S^{2\gamma}}$.

ii. When the correlation is positive, the strategy increases by a fraction: $\frac{m\delta(a^2S^{(2\gamma+1)}+1)}{fa^2S^{2\gamma}}$.

iii. When the correlation is unity, the strategy increases by a fraction by $\frac{\delta\{\alpha^2S^{2\gamma+1}\}}{fa^2S^{2\gamma}}$

iv. When the correlation is zero, the strategies are in equilibrium, $K(t) = K_{nc}^{*}(t)$.

This study optimizes the exponential utility of an investor's optimal strategy when the Brownian motions do not correlate and when they do correlate.

The study elaborates on the financial market assets, especially bonds and stocks where the price process of the risky asset followed the modified constant elasticity of variance (M.C.E.V) model and that of the risk-free asset the Ornstein-Uhlenbeck model.

Stochastic optimal control process was applied to obtain the corresponding Hamilton-Jacobi-Bellman (HJB) and using the maximum principle explicit solutions were obtained for the cases of when the Brownian motion do not correlate and when the Brownian motions do correlate.

REFERENCES

Merton, RC. 1969. Life time portfolio selection under uncertainty, the continuous time case. The Review of economics and statistics. 51(3):247-257.

Merton, RC. 1971. Optimum Consumption and Portfolio Rules in a Continuous-time model, Journal of Economic Theory. 3(4):373-413.

Karatzas, I., Lehoczky, JP. and Shreve, SE. 1987. Optimal portfolio and consumption decisions for a small investor on a finite horizon, SIAM Journal on Control and Optimization. 25(6):1557-1586.

Gao, J. 2009. Optimal investment strategy for annuity contracts under the constant elasticity of variance model, Insurance Mathematics and Economics. 45(1):9-18.

Liu, J., Bai L. and Yiu, KC. 2012. Optimal Investment with a value-at-risk constraint, Journal of Industrial and Management Optimization[JIMO]. 8(3):531-457.

Oksendal, B. and Sulem, A. 2002. Optimal Consumption and Portfolio with both Fixed and Proportional Transaction Costs, SIAM Journal on Control of Optimization, 40:1761-1790.

Gu, A., Guo, X. and Li, Z. 2012. Optimal Control of excess-of-loss Reinsurance and Investment for Insurers Under Constant Elasticity of variance (CEV) model, Insurance Mathematics and Economics. 51(3):674- 684.

Lin, X. and Li, Y. 2011. Optimal Reinsurance and Investment for a Jump Diffusion Risk Process under the CEV Model, North American Actuarial Journal. 15:3, 417-431, DOI: 10.1080/10920277.2011.10597628.

Zhao, H. and Rong, X. 2012. Portfolio selection problem with multiple risky assets under the constant elasticity of variance model. Insurance Mathematics and Economics. Elsevier. 50(1):179-1.

Gu, M., Yang, Y., Li, S. and Zhang, J. 2010. Constant elasticity of Variance model for Proportional Re-insurance and Investment strategies, Insurance Mathematics and Economics. Elsevier. 46(3):580-587.

Chang, H., Rong, XM. and Zhao, H. 2013. Optimal investment and consumption decisions under the Ho-Lee interest rate model, WSEAS Transactions on Mathematics. 12:1065-1075.

Ihedioha, SA. 2016. Exponential Utility Optimization of an investors optimal portfolios, under constant elasticity of variance (CEV) model. International Journal of Innovative issue 7. ISSN: 23944404, 330335.

Ihedioha, SA., Oruh, BI. and Osu, BO. 2017. Effects of Correlation of Brownian Motions on an Investor's Optimal Investment and Consumption Decision under Ornstein-Uhlenbeck Model. Academic Journal of Applied Mathematical Sciences. 3(6):52-61.

Ihedioha, SA. 2017. Investor's Power Utility Optimization with Consumption, Tax, Dividend and Transaction Cost under Constant Elasticity of Variance Model. Asian Research Journal of Mathematics. 4(2):1-12.

Heath, D. and Platen, E. 2002. Consistent pricing and hedging for a modified constant elasticity of variance. Quantitative Finance. 2:549-467. https://github.com/Dmitrymuravey/MCEV benchmark.

Muravey, D. 2017. Optimal investment problem with M.C.E.V model: Closed from solution and application to the algorithmic trading Department of Probability Staklov Mathematical Institute RAS Moscow Russia. arXiv:1703.01574v3[q-fin.MF] 10th July 2018.

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